Modulational instability of an axisymmetric state in a two-dimensional Kerr medium

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We consider spatial evolution of small perturbations of an axisymmetric state in a stationary two-dimensional medium with attractive cubic nonlinearity. For the unperturbed state, we find a one-parameter family of exact weakly localized solutions. The perturbation is expanded over angular harmonics, and its growth along the radial coordinate is then considered. In contrast to the well known case of the one-dimensional modulational instability, the integral gain of the radially growing perturbations converges. It is calculated in the adiabatic approximation, which is valid when the amplitude A of the unperturbed state and the azimuthal "quantum number" of the perturbation are both large. In this approximation, the integral gain does not depend upon m, and it increases linearly with A.

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Consideration of axisymmetric solutions (vortices) to the two-dimensional (2D) nonlinear Schrödinger (NLS) equation, also known as the Ginzburg-Pitaevsky equation, was begun long ago in relation to superfluidity [1, 2], as well as in the context of nonlinear optics [3]. Recently, this general problem has gained a renewed interest [4] stimulated by the experimental observation of stable optical vortices in a self-defocusing Kerr medium [5].

Stable optical vortices in the form of a "hole" (a two-dimensional dark soliton) exist in nonlinear media with repulsive nonlinearity. In a medium with attractive nonlinearity, it is natural to expect that axisymmetric states (at least those sufficiently slowly vanishing at infinity) will be subject to modulational instability (MI). Analysis of MI is one of the fundamental issues in various physical applications, especially in nonlinear optics [6]. The objective of this work is MI analysis in the axisymmetric case.

We start with the general 2D wave equation with the attractive Kerr (cubic) nonlinearity:

$$u_{tt} - u_{xx} - u_{yy} - |u|^2 u = 0 . \tag{1}$$

We are interested in solutions of the form

$$u(r, \phi; t) = U(r, \phi)\, e^{-i\omega t + ikr} , \tag{2}$$

where k is an arbitrary radial wave number, r and ϕ are the polar coordinates, the function U is assumed to be slowly varying in comparison with $\exp(ikr)$, and $\omega^2 = k^2$. Substituting Eq. (2) into Eq. (1) and omitting the terms U_{rr} and $r^{-1}U_r$, which are much smaller than the other terms due to the assumed slow dependence of U upon r, we obtain the following radial NLS equation:

$$2iV_\rho + i\rho^{-1}V + \rho^{-2}V_{\phi\phi} + |V|^2 V = 0 , \tag{3}$$

where $V \equiv k^{-1}U$, $\rho \equiv kr$. In what follows below, we are interested in the development of MI along the radial coordinate ρ. It may be relevant to mention that this problem has a certain similarity to the stability analysis for an expanding cylindrical combustion front in a gas. It was found that the cylindrical flame's instability grew with time not exponentially, but as a certain power of time (see the original work [7] and the later one [8]). However, unlike that problem, ours is time independent.

Equation (3) has a one-parameter family of exact axisymmetric solutions

$$V_0(r) = A\,\rho^{-1/2 + iA^2/2} , \tag{4}$$

A being an arbitrary real parameter (the amplitude of the solution). The solution (4) is weakly localized, in the sense that $|V|^2 \sim \rho^{-1}$ at $\rho \to \infty$. However, its total energy

$$2\pi \int_0^\infty |V|^2 \rho\, d\rho$$

strongly (linearly) diverges at $\rho \to \infty$.

The solution (4) is irrelevant at very small ρ, where it diverges. However, this is not essential for this work, where we will study evolution of disturbances propagating from the center to the periphery. Actually, when the local amplitude of the solution becomes (at small ρ) very large, the cubic term in the underlying equations must be replaced by a saturable nonlinearity, which will check the growth of the amplitude.

A perturbed solution is sought in the form

$$V = V_0(\rho)(1 + a)e^{i\theta} , \tag{5}$$

where V_0 is the unperturbed solution (4), a and θ being real perturbations of the amplitude and phase. The linearized equations for these functions are

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$$2a_\rho + \rho^{-2}\theta_{\phi\phi} = 0, \quad (6)$$

$$-2\theta_\rho + 2A^2\rho^{-1}a + \rho^{-2}a_{\phi\phi} = 0. \quad (7)$$

In what follows below, it will be convenient to use the standard trick, searching for eigenmodes of the real linear system of Eqs. (6) and (7) in the complex form

$$a(\rho,\phi) = b(\rho)\, e^{im\phi}, \quad (8)$$

$$\theta(\rho,\phi) = \eta(\rho)e^{im\phi}, \quad (9)$$

where m is an arbitrary integer. Insertion of Eqs. (8) and (9) into Eq. (6) allows one to exclude $\eta(\rho)$ in favor of $b(\rho)$: $\eta = 2m^{-2}\rho^2 b'$, the prime standing for differentiation in ρ. Finally, Eq. (7) yields an equation for the amplitude $b(\rho)$

$$b'' + 2\rho^{-1}b' + \tfrac{1}{4}m^2\rho^{-4}\left(m^2 - 2A^2\rho\right) b = 0. \quad (10)$$

Here it is relevant to come back to Eq. (3) and notice that we have omitted some small terms when deriving it from Eq. (1). Obviously, Eq. (10) is meaningful if the term $\rho^{-2}V_{\phi\phi}$ in Eq. (3), which gives rise to the terms in Eq. (10) that depend upon m^2, is much larger than the omitted terms. A simple consideration shows that this condition is equivalent to the inequality $m^2 \gg 1$, which will be assumed to hold. Since the main result to be obtained below, Eq. (14), does not actually depend upon m^2, this inequality will not really hurt the applicability of the results. On the other hand, the fact that they apply (at least, formally) only to large m^2 but do not depend upon m^2 implies that the results will be pertinent for a case when the initial disturbance is a short-wave one, being rich in higher harmonics.

Generally, Eq. (10) cannot be solved in standard functions. However, an approximate solution can be found, in the adiabatic approximation, as follows:

$$b(\rho) = \exp\left(\int \gamma(\rho)d\rho\right), \quad (11)$$

where it is assumed that

$$|d\gamma/d\rho| \ll \gamma^2. \quad (12)$$

The conditions necessary for Eq. (12) to hold will be considered in detail below. Then, Eq. (11) yields

$$\gamma = \rho^{-1}\left(\sqrt{1+\tfrac{1}{2}Am^2\rho^{-1} - \tfrac{1}{4}m^4\rho^{-2}} - 1\right) \quad (13)$$

(there are two two different branches of γ; here we chose the one that can give rise to an instability).

The approximate expression (11) for the perturbation amplitude $b(\rho)$ describes growth of the perturbation, i.e., development of the MI, as long as γ given by Eq. (13) remains real and positive. According to Eq. (13), this holds at $\rho > \rho_0 \equiv m^2/2A$. Then, the *integral instability gain* (IIG) can be defined as follows:

$$\Gamma(m^2, A) \equiv \int_{\rho_0}^{+\infty} \gamma(\rho)\, d\rho. \quad (14)$$

A principal difference of the considered axisymmetric problem from the traditional 1D problem is the fact that here the IIG converges, while in 1D it obviously diverges at $z \to \infty$, z being the propagation distance.

It is necessary to stress that the convergence of the IIG is not an artifact of the approximation employed. Indeed, a straightforward asymptotic analysis of Eq. (10) at $\rho \to \infty$, without using the approximation (11), shows that the perturbation *never* grows in this limit. This is a drastic difference from the above-mentioned instability problem for the expanding cylindrical flame [7], where the perturbation slowly grows at $\rho \to \infty$, that is equivalent to a logarithmic divergence of the corresponding IIG. It is also relevant to emphasize that, although we have explicitly considered only the case of large m^2, the IIG for small m^2 does not diverge either. The simplest way to check this is to notice that Eqs. (6) and (7) without the ϕ derivatives ($m = 0$) do not have solutions growing at $\rho \to \infty$.

Now, it is relevant to obtain conditions which provide for the underlying inequality (12) to hold. First of all, this inequality does not hold at $\rho \to \infty$, when Eq. (13) yields $\gamma \approx \frac{1}{4}Am^2\rho^{-2}$. However, when analyzing this asymptotic expression, it is straightforward to see that the inequality (12) still holds at $\rho \ll Am^2$. On the other hand, the convergent integral (14), with $\gamma(\rho)$ as per Eq. (13), is dominated by a contribution from $\rho \sim m^2/A$. Confronting these two ranges of ρ, one concludes that the adiabatic approximation remains relevent for calculation of the IIG provided that $A^2 \gg 1$. The meaning of this condition is very simple: the adiabatic approximation is legitimate if the dimensionless amplitude A of the unperturbed solution is large.

The approximation is not valid either in a vicinity of the above-mentioned point ρ_0, at which $\gamma(\rho) = 0$. However, it is easy to check that the validity of the approximation is restored at $\rho - \rho_0 \gg mA^{-3/2}$. Comparing this with the estimate given above, one concludes that this restriction is immaterial due to the conditions adopted, $A^2 \gg 1$ and $m^2 \gg 1$.

Let us come back to calculation of the IIG. Inserting Eq. (13) into Eq. (14), one can perform the integration to arrive at the expression $\Gamma(m^2, A) = A\tan^{-1}(A/2) - \ln(1 + A^2/4)$, which can be further simplified with regard to the condition $(A/2)^2 \gg 1$:

$$\Gamma(m^2, A) = \frac{\pi}{2}A - 2\ln\frac{A}{2}. \quad (15)$$

Notice that the expression (15) does not actually depend upon the azimuthal "quantum number" m (although this expression is relevant, as was explained above, only for large m^2). Thus, in the approximation considered, the integral MI gain takes the same value for all the values of m. Having arbitrary initial perturbations $a^{(0)}(\phi)$ and $\theta^{(0)}(\phi)$ at a certain point $\rho^{(0)}$, one should decompose them over the set of the eigenmodes $\exp(im\phi)$, considering the evolution of each component separately as was done above.

It is noteworthy to mention that IIG (15) grows approximately linearly with amplitude A of the unperturbed solution (4). This means that the actual amplification factor of the perturbation produced by the MI, $\exp\Gamma(A)$, grows exponentially with A,

$$e^{\Gamma(A)} = (A/2)^{-2}e^{\pi A/2}. \tag{16}$$

In conclusion, in this work we have considered the evolution of small perturbations along the radial coordinate on the background of an axisymmetric state in a two-dimensional stationary medium with the Kerr attractive nonlinearity. The unperturbed state was found in an exact form. In the adiabatic approximation, valid when both the amplitude of the unperturbed state and the azimuthal "quantum number" m are large, we have calculated the integral modulational instability gain for azimuthal eigenmodes of the perturbations. The integral gain does not depend upon m, and it is a linearly growing function of the unperturbed state's amplitude. However, the integral gain converges in contrast with the genuine 1D case.

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